Vortrix Algebra

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ABSTRACT

Vortrix Algebra is an improved vector algebra which provides complete and invertible forms of vector multiplication and division. Vortrix Algebra is derived from simple arithmetic which shows that the products of even numbers of vectors (even products) result in matrices, odd products result in vectors. This disambiguation between vectors and matrices resolves some of the anomalies and ambiguities inherent in modern mathematics. Such resolutions include the first ever vector products which provide for non-ambiguous inversion, the first ever non-imaginary solution to the square root of -1 which results in a pair of real operators to replaced the single imaginary complex operator (The Geometric Algebra definition is shown to be erroneous), powerful vector trigonometric functions such as sine, cosine and tangent, the ability to take limits of complex vector expressions providing for full vector calculus (Vortrix Calculus) to supplement the present “Partial” vector calculus which is based on partial derivatives.

Although the goal was to provide a more complete set of vector operators to support a more complete vector calculus; the development necessary to achieve those goals provided a step toward the unification of algebraic systems. This paper demonstrates that Vortrix Algebra is a superior alternative to all other vector algebras including Geometric Algebra and its derivatives. Secondly, since Vortrix Algebra is developed from standard arithmetic algebra, a Vortrix Algebra system of zero dimensions is consistent with arithmetic algebra. Finally, because real solutions are provided for the once “undefined” square root of -1, there is no further need for complex algebra or any other algebra requiring imaginary numbers. The benefit of replacing complex algebra with Vortrix Algebra is that engineers need only master one form of algebra for all applications. Furthermore, engineers can use real dimensions instead of imaginary dimensions. Vortrix Algebra should not be considered the final step of algebraic unification as there are anomalies of mathematics that still remain.

The incomplete sections of this paper are marked <INCOMPLETE>

1 Background (Prior Art)

The development of a more complete vector calculus requires a vector algebra with a complete and invertible set fundamental operators (addition, subtraction, multiplication and division). These operators are the foundation of all higher order mathematics. For example, they are necessary to compute the limits of any kind of vector expression; consequently, vector limits are necessary for Vector Calculus. This section discusses legacy vector systems, including legacy vector algebra (LA) and Geometric Algebra (GA) in terms of the completeness of operators

Prior to delving into these systems, it is appropriate to discuss the relationship between symbolic and numerical operators first.

1.1 Symbolic and Real Operators

There are instances throughout science and engineering where a vector appearing on both sides of a derivation are "divided" out and the result is experimentally shown to be sound. For example, the following arbitrary expression is given.

$KV\mathbf{B} = m\mathbf{B}$ (Vectors are identified by **bold** type)

The expression is reduced by dividing both sides by the vector B as follows

$KV = m$

This division is both symbolic and trivial. It is symbolic because real values are not actually being divided using a real numerical operator; rather, the division is Symbolic. It is trivial because it is obvious that anything divided by itself results in unity. This would be true even if the corresponding real operator did not exist. This is an example of the divergence between symbolic and real operators.

Next consider resolving the following:

$$\frac{(\mathbf{AB})}{\mathbf{B}} = ?$$

Symbolically the answer should be A; however, substituting real values for A and B produces a numeric result which does not contain enough information to non-ambiguously divide by B. Thus there is no real divide operator to support the symbolic divide and the existing numerical operator is insufficient to support the symbolic multiply. This shortcoming is exposed in the next section. In contrast, the symbolic expression contains sufficient information about the contents of the numerator to allow trivial symbolic division by B. This shows a clear divergence between symbolic and real operators.

Finally consider

$$\frac{(\mathbf{AB})}{\mathbf{C}} = ?$$

Since there are no trivial symbolic reductions available, the above expression can only be reduced by real operators which involve real values for A, B and C and the expression solved with real numerical multiply and divided which result in a real output. Prior Art shows no suitable vector system which provides a real vector divide, nor a complete vector multiply that retains sufficient information to support inversion.

The divergence between Symbolic and Real operators must be resolved. The following sections explore the divergence of popular legacy vector algebraic systems.

1.2 Legacy Vector Algebra (LA)

The term Legacy Vector Algebra (LA) is the term used in this work to describe the standard vector algebra that engineers are familiar with which was developed by Heaviside. The purpose of embellishing it with "Legacy" is to distinguish it from the generic term vector algebra.

Consider two vectors written in algebraic format

$$\mathbf{A} = Ax + Ay$$
$$\mathbf{B} = Bx + By$$

In the above, the values Ax and Bx represent quantities of length in the x directions, while Ay and By represent quantities of length in the y direction.

Before continuing, it is time to introduce definitions that are used in the remainder of this work. A vector, by standard definition, has both magnitude and direction. Sometimes it is desired to isolate the magnitude and direction of a vector for various reasons. The first definition is the magnitude of a vector which is defined as follows

$$|\mathbf{B}| = mag(\mathbf{B}) = \sqrt{BxBx + ByBy} \quad \text{The magnitude of a vector}$$

In the above, the left two forms are the symbolic representations while the form on the right is the real or numerical construct. As in legacy systems, the magnitude of a vector is a scalar quantity.

The second definition is the direction of a vector (direction vector for short) which is defined as follows

$$\hat{\mathbf{B}} = dir(\mathbf{B}) = \measuredangle\mathbf{B} = \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{Bx + By}{\sqrt{BxBx + ByBy}}$$

In the above, the left four forms are the acceptable symbolic representations and the right most expression is the real or numerical construct.

The direction vector is a vector of unit length (length =1) which points in the same direction as the original vector. It is sometimes called a unit vector or a unit direction vector.

A vector can be reconstituted by multiplying the magnitude and direction vector:

$$\mathbf{B} = |\mathbf{B}|\hat{\mathbf{B}} = mag(\mathbf{B})\ dir(\mathbf{B}) = |\mathbf{B}|\measuredangle\mathbf{B}$$

Now that the initial definitions are out of the way, it is time to resume the exploration of legacy vector products.

The product of vectors A and B is essentially the multiplication of the components for Vectors A and B which results in the following.

$$\mathbf{AB} = AxBx + AxBy + AyBx + AyBy$$

The outer two terms are identical to the legacy Dot product and rewritten as

$$\mathbf{AB} = \mathbf{A} \bullet \mathbf{B} + AxBy + AyBx$$

The legacy Dot product is defined by the following expression where theta (θ) is the angle between vectors A and B (angleAB).

$$\mathbf{A} \bullet \mathbf{B} = |\mathbf{A}||\mathbf{B}|\,\mathrm{Cos}\,\theta$$

The remaining two terms look almost like the traditional cross product which is defined for 2 Dimensional (2D) vectors as:

$$\mathbf{A} \times \mathbf{B} = (AxBy - AyBx)\hat{\mathbf{n}} \qquad \mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}|\,\mathrm{Sin}\,\theta\hat{\mathbf{n}}$$

Where the magnitude of the cross product is the area of the parallelogram formed by vectors A and B.

There are three problems with this definition.

The first problem is the minus sign which is inconsistent with the arithmetic result. It is certain that somewhere in the history of the development of vector algebra it was decided that multiplying an X component by a Y component yields a positive result, while multiplying a Y component by an X component yields a negative result. Although this choice may be completely arbitrary, it is a very useful mathematical construct. It allows cross products to be zero when vectors A and B are parallel. The ramification of this choice being that vector algebra is no longer commutative; rather it is anti-commutative which means **A**x**B** =-**B**x**A**. This choice eliminates the possibility of using arithmetic division to provide an inverse process. For the remainder of this paper this is referred to as the cross product sign convention. The development of Vortrix Algebra arrives at an anti-commutative cross product convention which is similar to the legacy version.

A second problem is the structure of the cross product result. The result is a vector that is normal to the input vectors A and B. In 2-dimensional (2D) space, there is no definition of a 3rd dimension and so this practice requires an imaginary construct. It is the intent of this work to eliminate all imaginary constructs and replace with real constructs. Furthermore, for 4D or higher dimensional systems, every other dimension is orthogonal to the plane formed by the dimensions X and Y (XY plane) and so trying to define a normal to an XY plane is completely ambiguous. For cross products in 3D systems it is just coincidence that there is only one orthogonal dimension to each pair of dimensions; therefore, only in a 3D system are cross product "normals" valid. Although this system has great utility, it is problematic for systems other than 3D.

A third problem with the traditional cross product is the units. The cross components AxBy and AyBx have the units of area which is inconsistent with the definition of the cross product which results in a vector of length. To be fair, the definition claims that the length of the vector is equal to the area. To define area in terms of length may be a useful, but sloppy, work-around in the engineering field; however, to the rigorous field of mathematics, this transgression should have been resolved generations ago.

In any event the legacy vector product is written as

$$\mathbf{AB} = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}|(\text{Cos}\,\theta + \hat{\mathbf{n}}\,\text{Sin}\,\theta)$$

To demonstrate that the legacy product does not provide sufficient information to invert the product of A and B, consider the product AB with vector A=(4,3) and vector B=(7,2), thus:

$$\mathbf{AB} = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \times \mathbf{B} = 34 - 13\hat{\mathbf{n}}$$

The challenge is to divide (4,3) into 34-13n to result in (7,2) without any prior knowledge that the divisor is contained in the dividend. To reduce this logically, the 34 is from the dot product which is defined as follows

$$\mathbf{A} \bullet \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos(\theta)$$

The normal results from the cross product which is

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin(\theta)\hat{\mathbf{n}}$$

The simplest thing that can be done at this time is to divide by the magnitude of (4,3)

$$\frac{|\mathbf{A}||\mathbf{B}|\cos(\theta)+|\mathbf{A}||\mathbf{B}|\sin(\theta)\hat{\mathbf{n}}=34-13\hat{\mathbf{n}}}{\sqrt{4^2+3^2}}$$

Using R to represent the result since there is no foreknowledge of the components of the original product, results in

$$|\mathbf{R}|\cos(angleR)+|\mathbf{R}|\sin(angleR)\hat{\mathbf{n}}=6.8-2.6\hat{\mathbf{n}}$$

The value angleR is determined from

$$\frac{|\mathbf{R}|\sin(angleR)}{|\mathbf{R}|\cos(angleR)}=\frac{-2.6}{6.8}=\tan(angleR)$$

$$-20.9245\deg=\tan^{-1}(\frac{-2.6}{6.8})$$

Then logically the magnitude of the result R is

$$|\mathbf{R}|=\frac{6.8}{\cos(-20.9245)}=7.28$$

Checking this with the other

$$|\mathbf{R}|=\frac{-2.6}{\sin(-20.9245)}=7.28$$

It is clear that the length of the quotient is 7.28. The next step is to determine the direction of the quotient; this is the crux of the problem. The direction of the quotient is either the sum of, or the difference between, the angle of the divisor and the angle -20.9245 degrees. If the algorithm had foreknowledge that the divisor is the left or right factor of the dividend, then the choice of direction is simple. From a symbolic standpoint, it is easy to show that (AB)/B results in the vector A because the symbolic expression contains complete information of the original components of the dividend. This highlights the loss of information inherent in the Legacy Product.

The above ambiguity is analogous to the answer of the sqrt(4) which could either be 2 or -2 depending if the 4 was the product of -2,-2 or 2,2 or something else. Again, it would require foreknowledge or some other form of information passed along that would be sufficient to non-ambiguously invert the function.

Further, consider the non-trivial case of (AB)/C. As before, determining the magnitude of the quotient is easy; however, the direction of the quotient is again ambiguous. The direction of the quotient is either the direction of C + 20.9245 degrees or the direction of C – 20.9245 degrees. Because of the arbitrary nature of this case, foreknowledge is of no value.

Vortrix algebra develops a more robust vector product which supports a non ambiguous divide.

1.3 Geometric Algebra (GA)

Geometric Algebra (GA) is an alternative vector algebra which defines a vector product as

$$\mathbf{AB} = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \wedge \mathbf{B}$$

This is called the Geometric Product where the dot product is identical to the dot product of LA and is either referred to as the dot product or the inner product. The second term (A^B) is either referred to as the outer product or exterior product depending on author. For the remainder of this work, the term exterior product is used. The definition of exterior product from Wikipedia reads

"The exterior product of two vectors can be identified with the signed area enclosed by a parallelogram the sides of which are the vectors."

It is well known that any two vectors (that are not parallel) form the sides of a parallelogram; so where is the product?

If the input to the exterior product is a pair of vectors that form the sides of a parallelogram and the result of the exterior product is a pair of vectors that form the sides of a parallelogram then what was done?

Thus, GA solves the loss of information problem by performing no product at all. By simply retaining the input vectors intact has traded a real operation for a symbolic operation. The symbolic nature of the exterior product is demonstrated by the form of the exterior product which is called a "Bivector". A Bivector is just a container to hold the two input vectors. From an engineer's perspective, this is not a product; rather, it's an I.O.U. for a product.

The advantage is that the product of AB now retains full information about the original products allowing the trivial expression (AB)/B=A to be resolve unambiguously. This resolution is symbolic in nature because the exterior product is no more of a real product than LA; as such, it is still not possible to resolve the expression (AB)/C with real values.

From the perspective of a real numerical operator (where actual values are multiplied), the GA product is actually less of a real product than the LA product. To demonstrate this, consider that both LA and GA use the LA Dot product and Cross product; except that, GA jettisons the LA cross product vector, retaining only its magnitude to quantify the area of the parallelogram for the exterior product. The shortcomings of the GA exterior product are demonstrated in section 4.15.3.

2 Development of Vortrix Algebra in 2D

In order to develop a complete set of vector products (multiplication and division), it is best to begin by determining what is needed from the completely missing operator: Division. By examining arithmetic division, the properties and functionality are extrapolated to those desired for vector division.

Arithmetic division is essentially a ratio of two values which serves three fundamental purposes. First, it forms the inverse to multiplication. Secondly, it is useful to apportion things; for example, if we have 50 apples and 9 buckets, then we could put 5 apples in each bucket with 5 apples left over. Lastly, it provides a ratio that allows us to scale other values. For example, if an existing home of 3000 square feet cost $300,000 to build, and the architect is asked to give a “ball park” estimate for a 4000 square foot structure, he could simply scale the price using a ratio; that is 4000/3000*300,000 = $400,000.

The ability to scale something by a ratio requires general forms of both division and multiplication and is therefore the least trivial of the three applications. By studying the least trivial application of the most complex operator should yield the most complete understanding of what a vector product should be.

In arithmetic, the ratio of B/A represents a value that can scale another value by the proportion of B to A. If B is twice as large as A, then B/A represents a “transmutor” that will double anything that it is multiplied against; consequently, it will transmute A to B {B= (B/A) A}. The real question is, what does this transmutor look like if A and B are vectors? Extrapolating the arithmetic case, the ratio of vectors B/A must be able to transmute A to B (A(B/A)= B). Logically, the ratio B/A must be a construct that can both scale and rotate a vector multiplied against it. Presently, the only means by which vectors can be both scaled and rotated is with matrices. This is the first hint that vector products (a term used henceforth to include both vector multiplication and division) may result in matrices rather than the traditional forms found in legacy systems.

Since it has been reasoned that the product of two vectors must be able to rotate and scale a third vector, it is imperative to explore the means by which a product of two vectors could produce rotation and scaling. In this early stage, it is not yet known how vector products can produce rotation; however, vectors can be scaled with a scalar and there already exists a vector product that results in a scalar. This is where we begin.

From the LA product, the product of two parallel vectors results in a scalar which is primarily due to the LA dot product. The simplest form of parallel product is the product of a vector multiplied by itself (the square vector).

$$\mathbf{AA} = \mathbf{A}^2 = |\mathbf{A}|^2$$

This definition is sufficient as "Stepping-Stone" to enable the development of Vortrix Algebra. A more complete definition of the square vector is realized as Vortrix Algebra is developed further.

In order to develop a complete multiplication and division, an expression is considered were a multiplication is inverted with division. Given the multiplication of vectors A and B, B must be recovered when dividing the result of the multiplication by A. Expressing this concept algebraically gives:

$$\mathbf{B} = \frac{\mathbf{AB}}{\mathbf{A}}$$

Multiplying top and bottom of the right hand side (RHS) by vector A, then applying the definition of the square vector yields:

$$\mathbf{B} = \frac{(\mathbf{AB})\mathbf{A}}{|\mathbf{A}|^2}$$

Since the denominator results in a scalar, the exploration of vector multiplication and division reduces to understanding the triple vector product (AB)A. To avoid trivial results, the more general triple product (AB)C is considered instead.
Expanding for the 2 dimensional (2D) case yields the following terms:

(AxBx)Cx
(AxBx)Cy
(AxBy)Cx
(AxBy)Cy
(AyBx)Cx
(AyBx)Cy
(AyBy)Cx
(AyBy)Cy

Since these terms must resolve to a vector in the original space (X,Y), a means is required to determine what each term resolves to. The magnitude of each term is simply the arithmetic multiplication of the three factors. The question is: what dimension (X or Y) does the product of these results couple to? The answer is inferred from the original triple product B=(AB)A. Speaking only in terms of direction, in order to obtain a result in the direction of B, the product (AB) must apply a rotation to the vector A (input vector) which would rotate it from A to B. Thus any vector multiplied by AB will be rotated by the same amount which works out to the direction of B minus the direction of A. This is expressed arithmetically as:

$$\hat{\mathbf{B}} = \left(\hat{\mathbf{A}}\hat{\mathbf{B}}\right)\hat{\mathbf{A}} \Rightarrow (\measuredangle\mathbf{B} - \measuredangle\mathbf{A}) + \measuredangle\mathbf{A} \Rightarrow \measuredangle\mathbf{B}$$

This determines the output dimensions for some of the terms

(AxBx)Cx : x (zero rotation)
(AxBx)Cy : y (zero rotation)
(AxBy)Cx : y (rotate Cx from x to y)
(AxBy)Cy : ?
(AyBx)Cx : ?
(AyBx)Cy : x (Rotate Cy from y to x)
(AyBy)Cx :x (zero rotation)
(AyBy)Cy :y (zero rotation)

For the remaining two terms, it is required to understand the result that occurs when the product AB is transposed to BA such that

$$? = \left(\hat{\mathbf{B}}\hat{\mathbf{A}}\right)\hat{\mathbf{A}}$$

The transposition of AB negates the direction of the applied rotation which is shown arithmetically as:

$$\mathit{reflect}(\mathbf{B},\mathbf{A}) = (\hat{\mathbf{B}}\hat{\mathbf{A}})\hat{\mathbf{A}} \Rightarrow \mathit{dir}((\angle\mathbf{A} - \angle\mathbf{B}) + \angle\mathbf{A})$$

The resultant output direction is the reflection of B about A. Essentially the angle from B to A is added to the direction of A to produce a result which looks like B being reflected about A. This is shown in Figure 1.

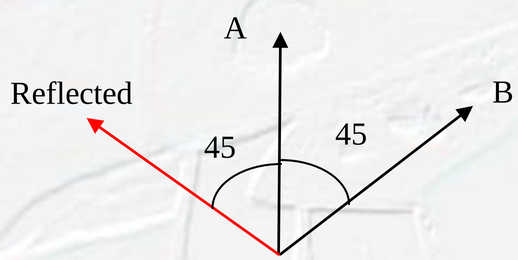

Figure 1: Reflection of B about A

A special case of reflection looks like negation, this occurs when the components A and B are perpendicular. For example, if A is along X (0 degrees) and B is along Y (90 degrees) then B reflected about A appears as simple negation of B.

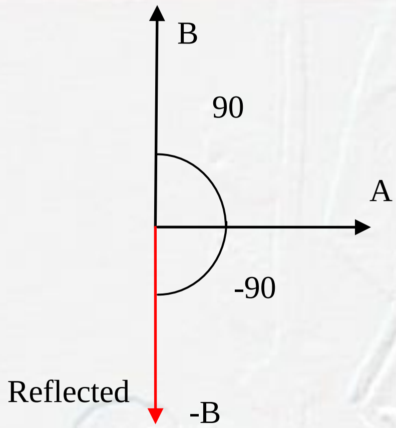

Figure 2: Reflection that Looks Like Negation

Expressing this special case arithmetically

$$-\hat{\mathbf{B}} = (\hat{\mathbf{B}}\hat{\mathbf{A}})\hat{\mathbf{A}} \Rightarrow (\angle\mathbf{A} - \angle\mathbf{B}) + \angle\mathbf{A} = -\angle\mathbf{B} \quad \textit{for } \mathbf{A} \perp \mathbf{B} \textit{ only}$$

From this, the remaining output dimensions are determined. The completed list is:

(AxBx)Cx : x
(AxBx)Cy : y
(AxBy)Cx : y
(AxBy)Cy : -x (Rotate Cy by 90 degrees (Ax to By = +90 rotation) y+90 =-x
(AyBx)Cx : -y (Rotate Cx by -90 degrees (Ay to Bx = -90 rotation) x-90 = -y
(AyBx)Cy : x
(AyBy)Cx :x
(AyBy)Cy :y

With a little bit of rearranging, it is quickly noted that the resulting terms form a matrix

$$(\mathbf{AB})\mathbf{C} = \begin{bmatrix} AxBx + AyBy & AyBx - AxBy \\ AxBy - AyBx & AxBx + AyBy \end{bmatrix} \begin{bmatrix} Cx \\ Cy \end{bmatrix}$$

Finally, by dropping the input vector (**C**), the proper result of the multiplication of Vectors A and B is revealed as a matrix.

$$\mathbf{AB} = \begin{bmatrix} AxBx + AyBy & AyBx - AxBy \\ AxBy - AyBx & AxBx + AyBy \end{bmatrix}$$

This result is consistent with the introduction which surmised that the products of two vectors cannot possibly exist in vector space. It is also interesting to note that a triple product was required to understand the true nature of a double product. It is important to note that there are twice as many terms resulting from this approach than occurred from the legacy investigation.

From the above products, a pattern emerges which suggests that double vector products (double products) result in a matrix and triple vector products (triple products) result in a vector. Consequently, odd products are vectors, even products are matrices.
To verify the completeness of this product, divide by A to see if B can be recovered.

$$\mathbf{B} = \frac{(\mathbf{AB})}{\mathbf{A}} = \frac{(\mathbf{AB})\mathbf{A}}{|\mathbf{A}|^2}$$

$$= \frac{1}{|\mathbf{A}|^2}\begin{bmatrix} AxBxAx + AyByAx + AyBxAy - AxByAy \\ AxByAx - AyBxAx + AxBxAy + AyByAy \end{bmatrix}$$

$$= \frac{1}{|\mathbf{A}|^2}\begin{bmatrix} AxBxAx + AyBxAy \\ AxByAx + AyByAy \end{bmatrix}$$

Then

$$= \frac{1}{|\mathbf{A}|^2}\begin{bmatrix} Bx(AxAx + AyAy) \\ By(AxAx + AyAy) \end{bmatrix}$$

Which is the same as

$$= \frac{1}{|\mathbf{A}|^2}\begin{bmatrix} Bx|\mathbf{A}|^2 \\ By|\mathbf{A}|^2 \end{bmatrix}$$

$$= \begin{bmatrix} Bx \\ By \end{bmatrix} = \mathbf{B}$$

Thus

$$\frac{(\mathbf{AB})}{\mathbf{A}} = \mathbf{B}$$

The next task is then to divide by B to recover A. Applying the same logic yields

$$\frac{(\mathbf{AB})}{\mathbf{B}} = \frac{(\mathbf{AB})\mathbf{B}}{|\mathbf{B}|^2} = |\mathbf{A}|\,reflect(\mathbf{A},\mathbf{B})$$

The above does not result in A. The result is a vector with the magnitude of A with the direction of A reflected about B. This results from the Cross Product sign convention that was developed. This convention eliminates the commutative property of multiplication and division making products left and right sensitive as well as order sensitive.

In order to recover A, the direction of the applied rotation from (AB) must be reversed. This is accomplished by transposing AB to arrive at:

$$\mathbf{A} = \frac{(\mathbf{BA})\mathbf{B}}{|\mathbf{B}|^2}$$

This is also accomplished by multiplying B from the left which follows:

$$\mathbf{A} = \frac{\mathbf{B}(\mathbf{AB})}{|\mathbf{B}|^2}$$

Because it is possible to multiply from the left or the right, it is also possible to divide from the left or the right. Furthermore, since it is possible to substitute division with an equivalent operation synthesized with multiplication. The question becomes, is right divide substituted with a left multiply or a right multiply? It turns out that equivalents are "opposite side" which means that the equivalent of left divide is synthesized using right multiply etc. This is demonstrated in the following.

Right division is represented by the symbol "/". Right division by B can be substituted with a left multiply by B and dividing by B^2.

$$\mathbf{A} = (\mathbf{AB}) / \mathbf{B} = \frac{\mathbf{B}(\mathbf{AB})}{|\mathbf{B}|^2}$$

Left Division is represented by the "\" symbols and its equivalent is Right Multiplication with a division by the square as shown in the following:

$$\mathbf{B} = \mathbf{A} \setminus (\mathbf{AB}) = \frac{(\mathbf{AB})\mathbf{A}}{|\mathbf{A}|^2}$$

Note: for the accompanying C# software tools, the right and left multiply operator is "*" while the right divide operator is "/" and the left divide operator is "%". The above expression is coded as A%(A*B) while the previous is (A*B)/B.

The property of right or left is here forward referred to as handedness. Again, the operator '/' is used to denote right division while the operator '\' denotes left division (% in the software). Right and left multiplication is inferred from which side the factor is on. The term AB can either be viewed as A left multiplied to B, or B right multiplied to A. The relationship between right and left operators is covered in more detail in a later section.

This solution for vector products represents the most complete form of vector multiplication and division to date. It unifies vectors, matrices, rotors and scalars into a coherent system suitable for modeling fluidic rotations (vortices). The name Vortrix is a contraction of Vortex and Matrix.

Because the solution demonstrates that matrices result from even vector products and vectors are the result of odd products, the widely held practice of considering vectors as simply just matrices of a single row, or column, is erroneous in spite of its usefulness.

This 2 dimensional result represents only the beginning. More phenomena are encountered as each system of higher dimension is explored.

3 Vortrix Algebra in 3D

Performing a triple product of 3 dimensional vectors (Ax+Ay+Az)(Bx+By+Bz)(Cx+Cy+Cz) results in the following 3x3 matrix:

$$[\mathbf{AB}] = \begin{bmatrix} AxBx + AyBy + AzBz & -AxBy + BxAy & -AxBz + BxAz \\ AxBy - BxAy & AxBx + AyBy + AzBz & -AyBz + ByAz \\ AxBz - BxAz & AyBz - ByAz & AxBx + AyBy + AzBz \end{bmatrix}$$

With additional "left-over" products that do not fit into the 3x3 Matrix shown below

(AyBz-AzBy)Cx
(AzBx-AxBz)Cy
(AxBy-AyBx)Cz

What are these extra products? Where do they go? How are they handled? These questions are not answered by any classical system. Since these extra components are the multiplication of three orthogonal dimensions (AxBy)Cz it is realized that they are volume terms. Because they are odd products, their proper disposition is vector space. This means that in order to support 3 dimensional (3D) products, 3D vectors must be upgraded to accommodate 3 dimensions of length and one dimension of volume. The volume dimension would be defined as dimension xyz and the vector would be written arithmetically as

$$\mathbf{A} = Ax + Ay + Az + Axyz$$

A four dimensional (4D) vector would have 4 lengths and 4 volumes and would be written arithmetically as

$$\mathbf{A} = Ax + Ay + Az + Aw + Axyz + Axyw + Axzw + Ayzw$$

Vortrix Algebra in 4D or higher is a later release and will not be discussed here further. Returning to the 3D system, the Axyz term is replaced by Av (v for volume) for the purpose of brevity.

$$\mathbf{A} = Ax + Ay + Az + Av$$

The inclusion of the volume term means that 3D vector products result in 4x4 matrices. Performing a triple product of the improved 3D vectors (Ax+Ay+Az+Av)(Bx+By+Bz+Bv)(Cx+Cy+Cz+Cv) results in the following [AB] matrix:

$$[\mathbf{AB}] = \begin{bmatrix} +AxBx+AyBy+AzBz+AvBv & -AxBy+AyBx-AzBv+AvBz & -AxBz+AyBv+AzBx-AvBy & -AxBv-AyBz+AzBy+AvBx \\ +AxBy-AyBx+AzBv-AvBz & +AxBx+AyBy+AzBz+AvBv & -AxBv-AyBz+AzBy+AvBx & +AxBz-AyBv-AzBx+AvBy \\ +AxBz-AyBv-AzBx+AvBy & +AxBv+AyBz-AzBy-AvBx & +AxBx+AyBy+AzBz+AvBv & -AxBy+AyBx-AzBv+AvBz \\ +AxBv+AyBz-AzBy-AvBx & -AxBz+AyBv+AzBx-AvBy & +AxBy-AyBx+AzBv-AvBz & +AxBx+AyBy+AzBz+AvBv \end{bmatrix}$$

For the purpose of computational efficiency, it is observed that the [AB] matrix resulting from an improved vector multiply can be represented by the terms in the first column. Each of the first column terms are represented by a label (M0 through M3 respectively). This means that an AB matrix can be stored in the same amount of space as an improved vector.

$$[\mathbf{AB}] = \begin{bmatrix} M0 = +AxBx+AyBy+AzBz+AvBv \\ M1 = +AxBy-AyBx+AzBv-AvBz \\ M2 = +AxBz-AyBv-AzBx+AvBy \\ M3 = +AxBv+AyBz-AzBy-AvBx \end{bmatrix}$$

The above 4 terms are expanded for matrix operations using the following template

$$[\mathbf{AB}] = \begin{bmatrix} M0 & -M1 & -M2 & -M3 \\ M1 & M0 & -M3 & M2 \\ M2 & M3 & M0 & -M1 \\ M3 & -M2 & M1 & M0 \end{bmatrix}$$

Note: As of this writing the behavior of the 3D system is not quite as expected. There is no evidence that it is wrong. It is more likely the case that my preconceptions were misguided. In fact, exhaustive numerical testing shows that all operations are fully invertible without loss of information or resulting in ambiguity. Furthermore, all identities which are shown in the following pages are exhaustively tested using the 3D system and all behave as expected. This note is added to keep the interested reader sharp to the possibility of error such that they can be identified and corrected ASAP so we can get on with the larger task at hand with the best possible tools.

4 Notations, Definitions and Identities

The following Notations, definitions and Identities apply to Vortrix System of all dimensions unless otherwise stated.

4.1 Dimensional Aggregation and Annihilation

In the arithmetic derivation of Vortrix algebra, it is observed that the multiplication of two vectors results in terms that are similar to the classical Dot Product and Cross Product. The "dot" terms are scalar and the "cross" terms having dimensions of either area or rotation depending upon context. From this observation it is concluded that the multiplication of parallel components Ax and Bx result in a value AxBx that is dimensionless. This dimensional annihilation is the same outcome that would be expected if Ax were divided by Bx. Conversely, if component Ax were multiplied by By, the result AxBy would have the units of area in the plane formed by X and Y. The terms Dimensional Aggregation is used to describe the result of a product of two dimensions which combine to form an aggregate of the dimensions (such as area). The term Dimensional Annihilation is used to describe the result of a product which has fewer dimensions than the input factors.

Because of Dimensional Aggregation and Annihilation, vector products produce a plurality of strange inter-dimensional results. These results are also called products; they are the product (results) of products. To disambiguate the term product, which refers to multiplication and division, from the term product which refers to the result of multiplication or division, the products which are the result of products are assigned a number. For example, the terms Ax and By are multiplied resulting in the 2-Dimensional term AxBy, this is called a 2^{nd} product. The product AxBx is a Zero product because it is dimensionless.

← Rotor → Flator

0	2	2	2
2	0	2	2
2	2	0	2
2	2	2	0

Vortrix 3D Matrix Products

← Linear → Volume

1	1	1	3

Vortrix 3D Vector Products

Zero Products are Dimensionless Scalar terms which either begin life as Scalars or are reduced to Scalars by Dimensional Annihilation. Zero products can only exist in the main diagonal of a matrix. Note: Vortrix Algebra "refines" the term Scalar in the next section

1^{st} Products (linears) are 1-dimensional terms such as Ax or AxByCx (Ax and Cx annihilate). 1^{st} Products either begin life as a value in a vector or are produced by annihilation. 1^{st} product terms can only exists in the linear section of vectors.
2^{nd} Products (rotors and flators) are 2-dimensional terms (AxBy) typically resulting from the product of two vectors (Double product) but can also be formed from annihilation of higher products. 2^{nd} Products are also called cross dimensional components (cross components or cross terms for short) and exist off the main diagonal of the matrix. There are two types of 2^{nd} Products called rotors and flators. Rotors exist in the linear section of the matrix and their principle function is to rotate the vector linears to other dimensions. Flators either inflate or deflate volumes. Flators are found in the Flator (green) sections of the matrix. The flators in the right column deflate volumes to linears. The flators in the bottom row inflate linears into volumes. It is important to highlight that in 3D systems there are only 3 unique second products. These 2^{nd} products are each repeated 4 times (as positive or negative) to become rotors or flators depending on the location in the matrix. Volume terms represent spatial volumes and exist only in the volume section of a vector. Volume Terms only exists in systems of 3-Dimensions and above

The 3D product yields a matrix with only 4 unique terms which might appear that something was lost; except, that each term is used 4 times in the matrix where the position in the matrix assigns a different "role". This demonstrates how the form of the result provides information beyond the scope of the simple one dimensional arithmetic operator from which it was developed. It is this type of information that is lost in legacy vector algebras because of improper or incomplete forms.

4.2 Scalars and Zero Products

In Vortrix algebra, scalars can only exist in the main diagonal of a matrix because the main diagonal of a matrix is the only place that can hold “dimensionless” numbers which are alternatively referred to as zero products. A zero product either begins life as a scalar or is produced from annihilation of higher products.

Note: This is the delineation point for the use of the word scalar, prior to this point the word scalar is used in the classical sense. From here forward the Vortrix definition of scalar is used.

An example of the proper representation of the scalar pi is:

$$\begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix}$$

In classical mathematical language, the above structure is called a “Scalar Matrix”. In Vortrix Algebra, a scalar can ONLY exists as a Scalar Matrix and so the term is redundant. Any of the following representations of a scalar matrix are allowed

$$\begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix} = [\pi] = \pi$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = [5] = 5$$

In future text, the term “Scalar Matrix” is sometimes used to remind the reader that a scalar is a matrix; however, the terms Scalar Matrix and Scalar are considered the same thing from here forward.

Because scalar matrices are left/right insensitive they are fully commutative. This property is heavily exploited in later sections of this paper

4.3 Bracket Notation and Matrix Delineation

To highlight a matrix in Vortrix Algebra, the elements that form a matrix are enclosed by square brackets as shown in the following examples

$$\mathbf{AB} = [\mathbf{AB}]$$
$$(\mathbf{AB})\mathbf{C} = [\mathbf{AB}]\mathbf{C}$$

The reason for the delineation is that matrices and vectors each have unique operators. An example is the matrix conjugate which is not available to vectors.

4.4 The Vortrix Dot and Cross Products

The matrix formed from the product of two vectors can be separated into two components. The components are the Vortrix Dot Product and the Vortrix Cross Product; both of which are supersets of the legacy vector products. The Vortrix Matrix is the sum of Cross and Dot products as shown

$$[\mathbf{AB}] = [\mathbf{A} \bullet \mathbf{B}] + [\mathbf{A} \times \mathbf{B}]$$

4.4.1 The Vortrix 2D Dot Product [A•B]

The Vortrix Dot Product for 2D and 3D systems are represented in the following diagrams. The 3D representation shows the Vortrix dot product in terms of the legacy Dot product in order to give perspective of how the old and the new relate:

$$[\mathbf{A} \bullet \mathbf{B}] = \begin{bmatrix} \mathbf{A} \bullet \mathbf{B} & 0 \\ 0 & \mathbf{A} \bullet \mathbf{B} \end{bmatrix} \text{ (for 2D only)}$$

$$[\mathbf{A} \bullet \mathbf{B}] = \begin{bmatrix} AxBx + AyBy & 0 \\ 0 & AxBx + AyBy \end{bmatrix} \text{ (2D, 3D \{with Z=0 v=0 for A and B\})}$$

4.4.2 The Vortrix 2D Cross Product [AxB]

The following are the Vortrix Algebra cross products for 2D and 3D systems. The 2D Vortrix Cross Product is represented in terms of the legacy cross product for the same reason stated for the 3D dot product. The purpose of the dot product inside the bracket of the 2D cross product is to remove the normal and preserve the proper sign of the legacy cross product.

$$[\mathbf{A} \times \mathbf{B}] = \begin{bmatrix} 0 & (\mathbf{B} \times \mathbf{A}) \bullet \hat{\mathbf{z}} \\ (\mathbf{A} \times \mathbf{B}) \bullet \hat{\mathbf{z}} & 0 \end{bmatrix} \textit{where } \hat{\mathbf{z}} = \hat{\mathbf{x}} \times \hat{\mathbf{y}} \text{ (for 2D only)}$$

$$[\mathbf{A} \times \mathbf{B}] = \begin{bmatrix} 0 & AyBx - AxBy \\ AxBy - AyBx & 0 \end{bmatrix} \text{ (2D, 3D \{with Z=0 v=0 for A and B\})}$$

4.4.3 Properties of the 2D Vortrix Cross and Dot Products

4.4.4 Vortrix 3D Dot Product

<UNFINISHED>

$$[\mathbf{A}\bullet\mathbf{B}]=\begin{bmatrix} \text{+AxBx+AyBy+AzBz+AvBv} & 0 & 0 & 0 \\ 0 & \text{+AxBx+AyBy+AzBz+AvBv} & 0 & 0 \\ 0 & 0 & \text{+AxBx+AyBy+AzBz+AvBv} & 0 \\ 0 & 0 & 0 & \text{+AxBx+AyBy+AzBz+AvBv} \end{bmatrix}$$

$$[\mathbf{A}\bullet\mathbf{B}]=[\text{M0= +AxBx+AyBy+AzBz+AvBv}]\ \text{(Short hand for 3D)}$$

4.4.5 Vortrix 3D Cross Product

<UNFINISHED>

$$[\mathbf{A}\times\mathbf{B}]=\begin{bmatrix} 0 & \text{-AxBy+AyBx-AzBv+AvBz} & \text{-AxBz+AyBv+AzBx-AvBy} & \text{-AxBv-AyBz+AzBy+AvBx} \\ \text{+AxBy-AyBx+AzBv-AvBz} & 0 & \text{-AxBv-AyBz+AzBy+AvBx} & \text{+AxBz-AyBv-AzBx+AvBy} \\ \text{+AxBz-AyBv-AzBx+AvBy} & \text{+AxBv+AyBz-AzBy-AvBx} & 0 & \text{-AxBy+AyBx-AzBv+AvBz} \\ \text{+AxBv+AyBz-AzBy-AvBx} & \text{-AxBz+AyBv+AzBx-AvBy} & \text{+AxBy-AyBx+AzBv-AvBz} & 0 \end{bmatrix}$$

$$[\mathbf{A}\times\mathbf{B}]=\begin{bmatrix} \text{M0}=0 \\ \text{M1}=\text{+AxBy-AyBx+AzBv-AvBz} \\ \text{M2}=\text{+AxBz-AyBv-AzBx+AvBy} \\ \text{M3}=\text{+AxBv+AyBz-AzBy-AvBx} \end{bmatrix}\quad \text{(Short hand}$$

4.4.6 Properties of 3D Dot and Cross products

<UNFINISHED>

4.5 Vector Transpose and the Matrix Conjugate [AB]=[BA]*

Transposing the vectors of a vector multiply inverts the direction of matrix rotation. This is called Vector Transpose and is demonstrated by the following:

$$[\mathbf{BA}]=[\mathbf{B}\bullet\mathbf{A}]+[\mathbf{B}\times\mathbf{A}]=[\mathbf{A}\bullet\mathbf{B}]-[\mathbf{A}\times\mathbf{B}]$$

The end result of vector transpose is the negation of the rotational (Cross) components. This result is consistent with classical vector products where transposing vectors negates the cross product but does not affect the dot product.

The next step is to define an operator which only negates the cross dimensional products of a matrix. Because this operation is similar to the complex conjugate of complex arithmetic it is therefore dubbed the matrix conjugate and defined as follows

$\left[\mathbf{BA}\right]=\left[\mathbf{AB}\right]^{*}=\left[\mathbf{A}\bullet\mathbf{B}\right]-\left[\mathbf{A}\times\mathbf{B}\right]$ The Matrix Conjugate or Vector Transpose

It is very important to highlight that there is no conjugation for odd products (vectors); only even products can be conjugated. This separates Vortrix algebra from complex or quaternion algebra, where conjugation can be applied to even or odd products because in those systems, the results of products are always the same construct as the factors. A further distinction between Vortrix and complex algebra is that in complex algebra, the transposition of the factors does not affect the result of a complex multiply

(A+jB)(C+jD)= (C+jD) (A+jB).

An astute reader will notice that the conjugation of the matrix arrives at the same result as matrix transpose. It is not proper to assume that identical results prove the operations are analogous. This would be the same thing as stating that multiplication and addition are analogous because 2*2 and 2+2 arrive at the same result.

$\left[\mathbf{BA}\right]=\left[\mathbf{AB}\right]^{T}\leftarrow$ Usage is discouraged

Although matrix transpose and vector transpose use the word “transpose,” it is enticing to consider that matrix transpose and vector transpose are analogous operations; however, these are just coincidences. The proper analogous operation to vector transpose is matrix conjugate and the use of matrix transpose is discouraged.

4.6 Left and Right Multiply

Because multiplication in Vortrix Algebra is not commutative, it becomes important to understand what happens when matrix [AB] is multiplied by vector C on the right or left. A right multiply occurs when vector C is juxtaposed on the right side of matrix [AB], resulting in [AB]C. A left multiply is defined when C appears at the left, resulting in C[AB]. Right multiply [AB]C is already understood. Using two transpositions, a left multiply is converted into a right multiply as shown.

$\mathbf{C}\left[\mathbf{AB}\right]=\mathbf{C}\left[\mathbf{BA}\right]^{*}=\left[\mathbf{BA}\right]\mathbf{C}$

In the first step shown above, the [AB] matrix is conjugated and the vectors are transposed which are operations that cancel each other out. In the second step, C is transposed and the conjugate is dropped which also cancel out. By comparing right and left multiply

$$[\mathbf{AB}]\mathbf{C} = \mathbf{C}[\mathbf{BA}]$$

The above is important because it shows that a left multiply can be replaced by an equivalent right multiply. Using this identity the effect of multiplying AB from the right and the left is compared

$$\mathbf{C}[\mathbf{AB}] = [\mathbf{BA}]\mathbf{C} = [\mathbf{AB}]^{*}\,\mathbf{C}$$

4.7 Square Vector Scalar Matrix [AA]

Because a vector multiplied by itself, is the product of two parallel vectors, there are no rotational components (2nd products) in the result. The result contains only zero products; as such, the result is a scalar matrix.

$$[\mathbf{AA}] = \begin{bmatrix} AxAx + AyAy & 0 \\ 0 & AxAx + AyAy \end{bmatrix} = \begin{bmatrix} \mathbf{A}\bullet\mathbf{A} & 0 \\ 0 & \mathbf{A}\bullet\mathbf{A} \end{bmatrix} = [\mathbf{A}\bullet\mathbf{A}] = \mathbf{A}^2 = |\mathbf{A}|^2$$

Note, the use of $[\mathbf{A}\bullet\mathbf{A}]$ *and* $|\mathbf{A}|^2$ are deprecated because they represent a loss of information. The representations $[\mathbf{AA}] = \mathbf{A}^2$ allow for full algebraic manipulation without loss.

4.8 Unity Scalar Matrix

Other Vortrix Expressions that result in a scalar matrices are [A/A] and [A\A] which result in an identity (or unity) matrix

$$[\mathbf{A}/\mathbf{A}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [1] = 1$$

Again, a numeric value in brackets is a shorthand representation of a scalar matrix.

4.9 Split Matrix Notation

The following notation demonstrates an alternative representation of a vector divide.
.

$$[\mathbf{A}/\mathbf{B}] = \frac{[\mathbf{A}}{1}\frac{1}{\mathbf{B}]} = \frac{[\mathbf{A}}{\mathbf{B}]}$$

The ambiguous divide operator (the horizontal line separating numerator and denominator) itself does not convey whether the divide is left or right; therefore, the brackets are retained to express the handedness. This is called split matrix notation because it looks like the product is being separated and commuted (moved). In operations involving Scalar Matrices there is a limited commutative ability which is detailed in chapter 4.13.

For the most part, this notation makes it easier to demonstrate certain Vortrix Algebraic manipulations as well as allowing a more granular discussion of right and left operations. Warning, this notation may incorrectly give the impression that [A/B] is a combination of a left multiply and right divide. Just remember that there is only one operator in a given pair of brackets.

This notation is helpful for highlighting the subtlety between reciprocals and equivalent operators which covered in the next section.

4.10 Reciprocals and Equivalents

Vortrix operators can have both equivalents and reciprocals. The equivalent operation performs the same function while the reciprocal performs the inverse. For example, left divide can be replaced with an equivalent operation which is implemented with a right multiply in conjunction with a scalar divide. This is contrasted by the reciprocal of left divide which is left multiply.

Equivalents are important because they allow division to be replaced with multiplication and left operators to be replaced with right operators. This allows direct comparison of operations and allows a single software operator (in this case right multiply was chosen) to implement all vector multiplications and divisions operators.

It was originally believed that the reciprocal of left divide was right multiply; however, the derivation of the matrix inverse demonstrated this to be wrong. Consider the inverse of the matrix [AB]

$$[\mathbf{AB}]^{-1} = \frac{1}{[\mathbf{AB}]}$$

The proper equivalent for the reciprocal of [AB] is to solve the following

$$[\mathbf{AB}][\mathbf{AB}]^{-1} = 1$$

The obvious answer is

$$[\mathbf{AB}]\frac{[\mathbf{BA}]}{\mathbf{A}^2\mathbf{B}^2}=1$$

Where the equivalent for the reciprocal [AB] is implemented with [BA] as shown

$$\frac{1}{[\mathbf{AB}]}=\frac{[\mathbf{BA}]}{\mathbf{B}^2\mathbf{A}^2}$$

From the above result, it was determined that the equivalent of a right divide can be generated from a left multiply and vice-versa. These equivalent identities are expressed in the following split matrix identities

$$\frac{1}{[\mathbf{A}}\equiv\frac{\mathbf{A}]}{\mathbf{A}^2}$$
$$\frac{1}{\mathbf{A}]}\equiv\frac{[\mathbf{A}}{\mathbf{A}^2}$$

For example, given arbitrary vectors A and X, dividing A to the right of X is the Equivalent operation to multiplying A to the left and dividing by A^2

$$[\mathbf{X}/\mathbf{A}]=\frac{[\mathbf{AX}]}{\mathbf{A}^2}$$

Now that the equivalents of right and left divide are known, the next step is to determine the reciprocals of right and left divide. Because the reciprocal of [AB] is

$$[\mathbf{AB}]^{-1}=\frac{1}{[\mathbf{AB}]}$$

And because A is on the left side for both the normal and reciprocal, then the reciprocal of left multiply is left divide. The identities for both left and right are show in the following identities

$$([\mathbf{A})^{-1}=\frac{1}{[\mathbf{A}}$$
$$(\mathbf{A}])^{-1}=\frac{1}{\mathbf{A}]}$$

To give credence to the notion that the reciprocal of right divide is right multiply, consider the following expression where X is right divided by A and then right multiplied by A.

$$[\mathbf{X}/\mathbf{A}]\mathbf{A} = ?$$

Replacing right divide by its equivalent left multiply results in the following expression which shows the vector A multiplied against X from opposite directions effectively canceling any rotation of X. The division by the Scalar [AA] cancels any magnitude effects.

$$\frac{[\mathbf{AX}]\mathbf{A}}{\mathbf{A}^2} = \mathbf{X}$$

Another example begins with the arbitrary expression

$$\frac{[\mathbf{AB}]}{[\mathbf{CD}]} = ?$$

If the numerator were right multiplied by E, then in order to keep the ratio the same, the denominator must also be right multiplied by E such that

$$\frac{[\mathbf{AB}]\mathbf{E}}{[\mathbf{CD}]\mathbf{E}} = ?$$

Right multiply of the denominator is the same as right divide of the numerator.

4.11 Scalar and Unity Reciprocals

The reciprocal operations discussed in the previous section are opposing operations using the same vector that result in Scalar Matrix of Unity Magnitude. Therefore, the more precise definitions of those reciprocals are Unity Scalar Reciprocals or more simply Unity Reciprocals.

Scalar Reciprocals are opposing operations and vectors that result in a scalar matrix which is not necessarily unity. The Scalar Reciprocal of the right multiplication of A is the left multiplication by any vector which is parallel to A, antiparallel to A, to include A. The Scalar Reciprocal of left divide by A is left multiply by any vector which is parallel or antiparallel to A to include A.

Unity Reciprocals are a subset of Scalar Reciprocals.

In the remainder of this text, if the word reciprocal is used without being pre-qualified by the word Unity or Scalar, its meaning should first be inferred from context; otherwise, Unity is assumed.

4.12 Reflection

Reflection occurs when the direction of a vector relative to another vector is negated. This is demonstrated in Figure 1 where vector B is reflected about A. Reflection of B about A is implemented in Vortrix Algebra by performing two same side products of A against B. For example, B is reflected about A when A is first right multiplied to B to form the [BA] matrix and then right multiplied again to form a triple product [BA] A. The magnitude of the resultant vector is |B||B||A| and the direction is reflect(B,A) which is defined as the direction of B reflected about A. The following is an abbreviated list of the possible ways to implement reflection. In the following definitions reflect(B,A) mean the resultant direction is B reflected about A. The magnitude is shown separately.

$$[\mathbf{A}\backslash\mathbf{B}]\mathbf{A} = \mathbf{A}[\mathbf{B}/\mathbf{A}] = |\mathbf{B}|\,reflect(\mathbf{B},\mathbf{A})$$

$$\hat{\mathbf{A}}[\hat{\mathbf{A}}\mathbf{B}] = [\mathbf{B}\hat{\mathbf{A}}]\hat{\mathbf{A}} = |\mathbf{B}|\,reflect(\mathbf{B},\mathbf{A})$$

$$\mathbf{A}[\mathbf{A}\mathbf{B}] = [\mathbf{B}\mathbf{A}]\mathbf{A} = |\mathbf{B}|\,\mathbf{A}^2 reflect(\mathbf{B},\mathbf{A})$$

$$\mathbf{A}\backslash[\mathbf{A}\backslash\mathbf{B}] = [\mathbf{B}/\mathbf{A}]/\mathbf{A} = \frac{|\mathbf{B}|}{\mathbf{A}^2} reflect(\mathbf{B},\mathbf{A})$$

4.13 Commutative Properties of Split Scalars Reciprocals

The products formed from Scalar Reciprocals such as [AA] and [A/A] result in scalars matrices which are left/right ambiguous and fully commutative. In certain cases, products formed from Scalar Reciprocals can retain commutative ability even when split. For example, if B is left multiplied by A, then right multiplied by A, then the scalar matrix formed by the reciprocals can be commuted out. For example [AB]A= [AA]B= B[AA].

Scalar Reciprocals are able to maintain the commutative ability only if they are applied sequentially; for example, [AB]A = [AA]B and [BA]/A= [A/A]B=B. The ability to commute vanishes if not applied sequentially; for example, (C[AB])A != [AA][CB]. The following shows a table of identities checked by computer to understand the nature of Split Scalar Reciprocals.

1. PASS [AA][BC]=(A[BA])C
2. FAIL [AA][BC]=A([BA]C)
3. PASS [AA][BC]=(A[BC])A
4. PASS [AA][BC]=A([BC]A)
5. PASS [AA][BC]=[BC][AA]
6. PASS ([A[B[C[DE])A]=[B[C[DE][AA]

7. FAIL ([B[A[C[DE])A]=[B[C[DE][AA]
8. FAIL ([B[C[A[DE])A]=[B[C[DE][AA]
9. FAIL ([B[C[D[AE])A]=[B[C[DE][AA]

According to the above results, as long as [A and A] are applied sequentially, the commutative property is retained. This is expressed as follows

$$(\Psi / \mathbf{A})\mathbf{A} = \mathbf{A} \setminus (\mathbf{A}\Psi) = [\mathbf{A} \setminus \mathbf{A}]\Psi = \Psi$$
$$\mathbf{A}(\mathbf{A} \setminus \Psi) = (\Psi\mathbf{A}) / \mathbf{A} = [\mathbf{A} / \mathbf{A}]\Psi = \Psi$$
$$(\mathbf{A} \setminus \Psi) / \mathbf{A} = \mathbf{A} \setminus (\Psi / \mathbf{A}) = \frac{\Psi}{\mathbf{A}^2}$$
$$(\mathbf{A}\Psi)\mathbf{A} = \mathbf{A}(\Psi\mathbf{A}) = [\mathbf{A}\mathbf{A}]\Psi = \mathbf{A}^2\Psi$$
$$\textit{where } \Psi \textit{ is arbitrary vortrix expression}$$

This Split Scalar Reciprocal Commutative Property is used in the next section to explore right and left divide.

4.14 Left and Right Vector Divide

It was demonstrated that vector multiply has right and left forms; therefore, vector divide should also have right and left forms. The notation for the two forms are shown as follows

$$\left[\mathbf{A} / \mathbf{B}\right] = \textit{right divide}$$
$$\left[\mathbf{A} \setminus \mathbf{B}\right] = \textit{left divide}$$

Using Split Matrix notation and the Commutative Property of Split Scalar Reciprocals (4.13), the above expression is converted into equivalent multiplies

$$[\mathbf{A} \setminus \mathbf{B}] = \frac{\mathbf{B}]}{[\mathbf{A}}\left(\frac{\mathbf{A}]}{\mathbf{A}]}\right) = \frac{[\mathbf{BA}]}{\mathbf{A}^2}$$

In the above, the denominator is right multiplied by A] which is same as right dividing by A. This closes the matrix in the denominator converting it to a scalar matrix which releases B. Then right multiplying by A completes the reciprocal operation forming [BA] in the numerator.

Shown again not using split matrix notation

$$[\mathbf{A} \setminus \mathbf{B}] = \left([\mathbf{A} \setminus \mathbf{B}] / \mathbf{A}\right)\mathbf{A} = \left(\frac{\mathbf{B}}{\mathbf{A}^2}\right)(\mathbf{A}) = \frac{[\mathbf{BA}]}{\mathbf{A}^2}$$

In the above, [A\B] is right divided by A and the right multiplied by A which is a reciprocal operation. The right divided by A form a reciprocal operation with the left divide by A in [A\B] which results in B divided by the scalar A^2 which can then be commuted out of the way (using ambiguous divide symbol). Now that B is free, the right multiply by A forms the result shown in the above, at right, which is the equivalent of [A\B].

For completeness the Reciprocal operations are performed in the opposite order

$$\left([\mathbf{A}\setminus\mathbf{B}]\mathbf{A}\right)/\mathbf{A} = \left(\mathbf{A}\setminus[\mathbf{A}\setminus\mathbf{B}]\mathbf{A}^2\right)/\mathbf{A} = \mathbf{A}^2\left(\mathbf{A}\setminus[\mathbf{A}\setminus\mathbf{B}]\right)/\mathbf{A} = \mathbf{A}^2\left(\frac{[\mathbf{A}\setminus\mathbf{B}]}{\mathbf{A}^2}\right) = [\mathbf{A}\setminus\mathbf{B}]$$

In the above, the right multiply is performed first. The right multiply by A is replaced by its equivalent which is left divide by A and multiply by Scalar A^2. The A^2 is then commuted out of the way to reveal that [A\B] is successively divided on the right and the left by A which is a reciprocal operation resulting in divide by A^2. The A^2 in the numerator cancels the A^2 in the denominator and the [A\B] matrix is returned.
The following identities are developed using the same techniques

$$[\mathbf{B}/\mathbf{A}] = \left(\frac{[\mathbf{A}}{[\mathbf{A}}\right)\frac{[\mathbf{B}}{\mathbf{A}]} = \frac{[\mathbf{AB}]}{\mathbf{A}^2}$$

$$[\mathbf{BA}]/\mathbf{A} = \frac{\mathbf{A}[\mathbf{BA}]}{\mathbf{A}^2} = \mathbf{B}$$

$$[\mathbf{B}/\mathbf{A}]\mathbf{A} = \frac{[\mathbf{AB}]\mathbf{A}}{\mathbf{A}^2} = \mathbf{B}$$

$$\mathbf{A}\setminus[\mathbf{AB}] = \frac{[\mathbf{AB}]\mathbf{A}}{\mathbf{A}^2} = \mathbf{B}$$

$$\mathbf{A}[\mathbf{A}\setminus\mathbf{B}] = \frac{\mathbf{A}[\mathbf{BA}]}{\mathbf{A}^2} = \frac{[\mathbf{AB}]\mathbf{A}}{\mathbf{A}^2} = \mathbf{B}$$

$$[\mathbf{A}/\mathbf{B}] = [\mathbf{B}\setminus\mathbf{A}]^*$$

$$[\mathbf{AB}]/\mathbf{A} = \frac{\mathbf{A}[\mathbf{AB}]}{\mathbf{A}^2} = \hat{\mathbf{A}}\left[\hat{\mathbf{A}}\mathbf{B}\right] = |\mathbf{B}|\,reflect(\mathbf{B},\mathbf{A})$$

$$\mathbf{B}\setminus[\mathbf{AB}] = \frac{[\mathbf{AB}]\mathbf{B}}{\mathbf{B}^2} = \left[\mathbf{A}\hat{\mathbf{B}}\right]\hat{\mathbf{B}} = |\mathbf{A}|\,reflect(\mathbf{A},\mathbf{B})$$

$$\mathbf{C}\setminus[\mathbf{AB}] = \frac{[\mathbf{AB}]\mathbf{C}}{\mathbf{C}^2}$$

$$[\mathbf{AB}]/\mathbf{C} = \frac{\mathbf{C}[\mathbf{AB}]}{\mathbf{C}^2}$$

4.15 The Square Root of -1 (Complex Operator)

The complex operator, which is symbolized by the bold letter **i**, is defined as any value, that when multiplied by itself results in -1. Thus

$$\mathbf{i}^2 = -1$$
$$\mathbf{i} = \sqrt{-1}$$

The above result is called an “imaginary” number because it cannot be represented by a real number. It’s called the complex operator because it is used in complex algebra which is an algebraic system composed of real and imaginary numbers.

Vortrix Algebra provides the first ever solution (actually 2) for the complex operator that is completely based on real numbers. Because it is no longer complex, it is given the more appropriate name: Rotation Operator.

The following two sections derived the two Plane Rotation Operators of Vortrix Algebra

Geometric Algebra claims to have a definition for the complex operator; however, in section 4.15.3 it is demonstrated to be erroneous.

4.15.1 The Vortrix Plane Rotation Operator (i)

The Vortrix Plane Rotation Operator (PRO) is a direct drop-in replacement for the legacy complex operator. The PRO is non-imaginary and is constructed completely from real numbers.

Begin by considering that the purpose of the complex operator was to provide a definition for the square root of -1. The value -1 is a scalar, and the only place in Vortrix Algebra for scalars is the main diagonal of a matrix; therefore, the value -1 can only exist as a scalar matrix as discussed in section 4.2. Thus the value -1 is really

$$[-1] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The above matrix represents an operator that inverts the direction of a vector (180 degree change in direction). This is identical to multiplying a vector by -1 in LA. According to the previous section, the square root of the above should produce a matrix that only changes the direction of a vector by 90 degrees or

$$\sqrt{[-1]} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This identity is verified by squaring the matrix using standard matrix multiply which returns the original [-1]. This the first ever real definition of what used to be called the complex operator **i**. Now, because the value is no longer undefined or imaginary, there is no further need for imaginary dimensions and the operator is properly renamed: The Plane Rotation Operator or just rotation operator for short. The symbol of a bold faced **i** is retained to represent the rotation operator (electrical engineers will still use the symbol j so as not to confuse the operator with current) .

$$\mathbf{i} = \sqrt{[-1]} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Consequently

$$\mathbf{i}^2 = \sqrt{[-1]}^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = [-1]$$

This operator simply rotates vectors through 90 degrees in the XY plane and can be developed by multiplying a unit direction vector in x with the unit vector in y to form a +90 degrees rotation matrix as shown

$$\mathbf{i} = [\hat{\mathbf{x}}\hat{\mathbf{y}}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Because this definition is in terms of orthogonal unit vectors, we can reduce the definition to just the cross product

$$\mathbf{i} = [\hat{\mathbf{x}} \times \hat{\mathbf{y}}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Again, to prove that it is a root of -1, it is squared to show that the result is -1.

$$\mathbf{i}^2 = [\hat{\mathbf{x}} \times \hat{\mathbf{y}}]^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = [-1]$$

4.15.2 The Conjugate Plane Rotation Operator (-i)

The square root of any number should have two solutions (roots); therefore, the square root of -1 should also have two roots. Just as the sqrt(4) has both -2 and +2 as roots, the sqrt(-1) should have **i** and –**i** as roots. This is shown as follows

$$\mathbf{i}^* = -\mathbf{i} = [\hat{\mathbf{y}} \times \hat{\mathbf{x}}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

And is shown to be a root of -1 by squaring it to arrive at -1

$$(-\mathbf{i})^2 = [\hat{\mathbf{y}} \times \hat{\mathbf{x}}]^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = [-1]$$

The reason why it is called the conjugate plane rotation operator is that it results from a vector transpose which is a matrix conjugate. A matrix conjugate is the negation of the cross product portion of a matrix. This is why it is correct to use either a negative sign or a matrix conjugate operator.

The primary different between **i** and –**i** is the direction of rotation.

4.15.3 The Geometric Algebra Complex Operator Fraud

Geometric algebra claims to have developed the proper meaning of the complex operator in their definition of what they call “2D Unit Pseudoscalar.” They define the complex operator **i** as follows

$$\mathbf{i} = \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$$

Where the symbols in the exterior product are the unit direction vector in the X direction and the unit direction vector in Y direction respectively. The two direction vectors form an orthonormal basis for a 2D system. To prove that this is the complex operator, it must be squared and the result must equal -1.

$$\mathbf{i}^2 = (\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})(\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})$$

The first question is: what operator lies between the exterior products? The ONLY possible operator is a dot operator otherwise transposing the products would invert the sign of the result and $\mathbf{i}^2$ would ambiguously result in both 1 and -1. So placing in the dot operator results in:

$$\mathbf{i}^2 = (\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) \bullet (\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})$$

Since the exterior products are parallel constructs of vectors, then any sane logical person would identify that the dot product of identical vector constructs can only ever be 1; therefore, this derivation is totally invalid and proves that their definition of the Pseudoscalar is indeed false.

A sane engineer would test the above by substituting (1, 0) for x and (0, 1) for y and then numerically evaluate to see if negative 1 is the result. That engineer would then wonder why the authors GA didn't do that in the first place. Soon that engineer would realize that the exterior product is not a real product and it is not possible to directly evaluate the above expression. If that engineer decided to substitute the legacy cross product for the exterior product then the result would be 1. The same answer the sane person reasoned out.

In the interest of being thorough, the derivation is continued.

The next part of their proof is to discard the parenthesis and operators for no apparent reason to arrive at:

$\mathbf{i}^2 = \hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{x}}\hat{\mathbf{y}}$ (Note: Vortrix algebra returns a [+1] for this)

By discarding the operators they are implying that both operators of the Geometric Products are in play; if this is the case, then why bother defining **i** in terms of the exterior product? By dropping the parenthesis, they imply that the order of operation is not relevant; however, this is erroneous since changing the order of the products of

$\mathbf{i}^2 = (\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) \bullet (\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})$ to $\hat{\mathbf{x}} \wedge (\hat{\mathbf{y}} \bullet \hat{\mathbf{x}}) \wedge \hat{\mathbf{y}} = 0$ results in zero.

At this point, there is sufficient error to end this derivation. The only reason for continuing is that the next steps are very entertaining. Continuing from:

$\mathbf{i}^2 = \hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{x}}\hat{\mathbf{y}}$

Remember, the above has to equal -1 for their definition to be correct. To achieve this end, they swap the inner two products which they claim negates the result. Then they compensate by applying the negative sign.

$\mathbf{i}^2 = -\hat{\mathbf{x}}\hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{y}}$

The first way to expose the error is to replace the parentheses and operators

$-(\hat{\mathbf{x}} \wedge \hat{\mathbf{x}}) \bullet (\hat{\mathbf{y}} \wedge \hat{\mathbf{y}}) = 0$

Zero is certainly not -1; furthermore, the swap of the center items was around a dot operator which does not incur a sign inversion.

For the sake of fairness we correct their definition of the complex operator (Pseudoscalar) to better support what they are trying to do. By discarding the operator '^' from the definition, they are no longer limiting the definition to the specific use of the exterior product. This corrected definition becomes

$$\mathbf{i} = (\hat{\mathbf{x}}\hat{\mathbf{y}})$$

Then the proof becomes

$$\mathbf{i}^2 = (\hat{\mathbf{x}}\hat{\mathbf{y}})(\hat{\mathbf{x}}\hat{\mathbf{y}})$$

Now we are no longer limited by a definition as to which operator must be used. Furthermore, by discarding the parentheses, we are no longer bound to a specific order of operation. This is more consistent with the steps they have taken in the derivation. Returning to their previous step

$$\mathbf{i}^2 = -\hat{\mathbf{x}}\hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{y}}$$

One possible order of multiplication is to multiply the center items first; this could only be an exterior product because the dot product would result in zero. Resolving the center product allows us to substitute the definition of **i** as follows:

$$\mathbf{i}^2 = -\hat{\mathbf{x}}(\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})\hat{\mathbf{y}} = -\hat{\mathbf{x}}(\mathbf{i})\hat{\mathbf{y}}$$

Because the **i** operator causes a plus 90 degree rotation to the **y**, then it is converted into a –**x** and the result is

$$\mathbf{i}^2 = -\hat{\mathbf{x}}(\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})\hat{\mathbf{y}} = -\hat{\mathbf{x}}(\mathbf{i})\hat{\mathbf{y}} = -\hat{\mathbf{x}}(-\hat{\mathbf{x}}) = 1$$

This contradicts the desired result of -1 and demonstrates that order of operation is critical in vector products.

Let's try again with a different order of operation by multiplying the outer pairs first

$$\mathbf{i}^2 = -(\hat{\mathbf{x}}\hat{\mathbf{x}})(\hat{\mathbf{y}}\hat{\mathbf{y}}) = -(\hat{\mathbf{x}} \bullet \hat{\mathbf{x}}) \bullet (\hat{\mathbf{y}} \bullet \hat{\mathbf{y}}) = -(1)(1) = -1$$

$$\mathbf{i} = \sqrt{-1}$$

At best, they have simply rediscovered that the imaginary complex operator '**i**' is the square root of -1. At worst, this is a fraud that uses sleight of hand trickery and misdirection to cause a symmetrical product $(\hat{\mathbf{x}}\wedge\hat{\mathbf{y}})(\hat{\mathbf{x}}\wedge\hat{\mathbf{y}})$ appear as an asymmetrical product (-1)(1) to obtain a negative result. The motive is to claim (albeit falsely) that GA is isomorphic to complex algebra so that they can usurp the capabilities of complex algebra. Without the capabilities of complex algebra, GA is pointless gibberish.

The proper definition of the complex operator is a construct of real values, that when multiplied by itself, results in negative 1. Sorry but (-1)(1) does not count because it is not symmetrical and $\mathbf{i}=\hat{\mathbf{x}}\wedge\hat{\mathbf{y}}$ is invalid because you can't substitute real numeric values for x and y and obtain a result that satisfies the requirement. To this date there are only three viable definitions, one is imaginary and two are real.

The legacy definition

$\mathbf{i}=\sqrt{-1}$ This is imaginary because it can't be expressed with real numbers

And now the two Vortrix definitions which are no longer complex

$\mathbf{i}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ The Vortrix Plane Rotation Operator

$-\mathbf{i}=\mathbf{i}^*=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ The Vortrix Conjugate Plane Rotation Operator

The legacy definition is undefined imaginary voodoo that can't be rationalized in an arithmetic processor (ALU or Calculator), the Vortrix definitions are standard ordinary matrixes well suited for any numerical processor that can add, subtract, multiply and divide.

Vortrix Algebra does not claim isomorphism with complex algebra; rather, Vortrix algebra claims superiority because it eliminates the imaginary nonsense and replaces it with two real operators called the Plane Rotation Operator and the Conjugate Plane Rotation Operator.

With this, Vortrix Algebra supersedes all other multidimensional algebras to include complex numbers and quaternions.

4.16 Properties of the 2D Vortrix

This section demonstrates the properties of the 2D Vortrix Matrix. Given a 2D [AB] matrix

$$[\mathbf{AB}]=\begin{bmatrix}AxBx+AyBy & AyBx-AxBy\\ AxBy-AyBx & AxBx+AyBy\end{bmatrix}$$

It is observed that the elements along the diagonal are all the same, and the elements off the main diagonal are negatives of each other. This means that the 2D matrix can be represented by the elements in the first column allowing a matrix to be stored in a compact form for computational efficiency. Assigning M0 to represent AxBx+AyBy and M1 to represent AxBy-AyBx, the Matrix can be stored as:

$$[\mathbf{AB}] = \begin{bmatrix} M0 = AxBx + AyBy \\ M1 = AxBy - AyBx \end{bmatrix}$$

The Matrix is then be reconstituted using the following

$$[\mathbf{AB}] = \begin{bmatrix} M0 & -M1 \\ M1 & M0 \end{bmatrix}$$

4.16.1 Scalar Magnitude of Matrix

The magnitude of the matrix is the RSS (root sum square) of the elements of the first column.

$$|\mathbf{AB}| = \sqrt{\sum_{n=0}^{nD-1} (Mn)^2}$$

4.16.2 Rotation Angle

The rotation angle of the matrix is the 2D Arctan of the first column (atan2)

$$\measuredangle AB = \arctan 2(M1, M0)$$

4.16.3 Power of Vortrix

The magnitude and rotation angle represent how the matrix affects other things that are multiplied against it. For example, a vector right-multiplied to this matrix will be rotated counter-clockwise by the rotation angle and its magnitude increase by the scalar magnitude. Another matrix multiplied to this will result in a new matrix with the product of the magnitudes and the sum of the rotation angles. Therefore, the square of the matrix results in a square of the magnitude and a doubling of the rotation angle. The logical progression takes on the following:

$$[\mathbf{AB}]^2 = |AB|^2 \measuredangle(2AB)$$

$$[\mathbf{AB}]^n = |AB|^n \measuredangle(nAB)$$

$$[\mathbf{AB}]^{1/2} = |AB|^{1/2} \measuredangle(AB/2)$$

$$[\mathbf{AB}]^0 = [1]$$

$$[\mathbf{AB}]^{-n} = |AB|^{-n} \measuredangle(-nAB)$$

The last identity shows a reversal of direction for negative exponents which was demonstrated earlier when the matrix reciprocal was derived

$$[\mathbf{AB}][\mathbf{AB}]^{-1} = 1$$

Note: when multiplying matrices, the magnitudes multiply and the rotation angles add.

Finally, if

$$[\mathbf{AB}]^0 = [1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then logically

$$[\mathbf{A} \bullet \mathbf{B}]^0 = [1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\mathbf{A} \times \mathbf{B}]^0 = [0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The above are important when exponents of Vortrix Matrices are considered.

4.16.4 Trigonometric Properties (revised v1.3)

The terms in the matrix produced by a Vortrix Vector Product can be defined in terms of sine and cosine:

$$[\mathbf{AB}] = \begin{bmatrix} |\mathbf{A}||\mathbf{B}|\cos(angleAB) & -|\mathbf{A}||\mathbf{B}|\sin(angleAB) \\ |\mathbf{A}||\mathbf{B}|\sin(angleAB) & |\mathbf{A}||\mathbf{B}|\cos(angleAB) \end{bmatrix}$$

And likewise for divide

$$[\mathbf{B}/\mathbf{A}] = \begin{bmatrix} (|\mathbf{B}|/|\mathbf{A}|)\cos(angleAB) & -(|\mathbf{B}|/|\mathbf{A}|)\sin(angleAB) \\ (|\mathbf{B}|/|\mathbf{A}|)\sin(angleAB) & (|\mathbf{B}|/|\mathbf{A}|)\cos(angleAB) \end{bmatrix}$$

Where angleAB is the direction of B – direction if A.

Note: These results and properties are for the presently selected rule set (Sign convention) .

The Dot product is then

$$[\mathbf{A}\bullet\mathbf{B}]=\begin{bmatrix}|\mathbf{A}||\mathbf{B}|\cos(angleAB) & 0\\ 0 & |\mathbf{A}||\mathbf{B}|\cos(angleAB)\end{bmatrix}$$

And the Cross product

$$[\mathbf{A}\times\mathbf{B}]=\begin{bmatrix}0 & -|\mathbf{A}||\mathbf{B}|\sin(angleAB)\\ |\mathbf{A}||\mathbf{B}|\sin(angleAB) & 0\end{bmatrix}$$

The above Cross and Dot products are defined only for multiplication. The Dot product is essentially the diagonal components resulting from a vector multiply and the Cross product is comprised of the off-diagonal elements (cross components). By defining matrix functions for dot and cross we achieve the same effect as follows.

$$[\mathbf{A}\bullet\mathbf{B}]=[\mathbf{AB}].dot()=\begin{bmatrix}|\mathbf{A}||\mathbf{B}|\cos(angleAB) & 0\\ 0 & |\mathbf{A}||\mathbf{B}|\cos(angleAB)\end{bmatrix}$$

$$[\mathbf{A}\times\mathbf{B}]=[\mathbf{AB}].cross()=\begin{bmatrix}0 & -|\mathbf{A}||\mathbf{B}|\sin(angleAB)\\ |\mathbf{A}||\mathbf{B}|\sin(angleAB) & 0\end{bmatrix}$$

The new functions can also be applied to division (or any other matrix result) to yield the division dot product and the division cross product as follows

$$[\mathbf{B}/\mathbf{A}].dot()=\begin{bmatrix}(|\mathbf{B}|/|\mathbf{A}|)\cos(angleAB) & 0\\ 0 & (|\mathbf{B}|/|\mathbf{A}|)\cos(angleAB)\end{bmatrix}$$

$$[\mathbf{B}/\mathbf{A}].cross()=\begin{bmatrix}0 & -(|\mathbf{B}|/|\mathbf{A}|)\sin(angleAB)\\ (|\mathbf{B}|/|\mathbf{A}|)\sin(angleAB) & 0\end{bmatrix}$$

The above Matrix functions are defined in the software and are the preferred method for separating matrices into dot and cross components. For brevity in documents the following compound operators are used

$$[\mathbf{B}/\mathbf{A}].dot() = [\mathbf{B}/\bullet\mathbf{A}]$$

$$[\mathbf{B}/\mathbf{A}].cross() = [\mathbf{B}/\times\mathbf{A}]$$

4.16.5 Vortrix Trigonometric Functions (New in V1.3)

This section defines a useful set of trigonometric functions based on the Properties explored in the previous section.

The first trigonometric function is sine. Sine is defined as the cross components of a vector divide. For the C# software that accompanies this document one would use either sin(B,A) or (B/A).cross().

$$\sin(\mathbf{B},\mathbf{A}) = [\mathbf{B}/\times\mathbf{A}] = [\mathbf{B}/\mathbf{A}].cross() = \begin{bmatrix} 0 & -(|\mathbf{B}|/|\mathbf{A}|)\sin(angleAB) \\ (|\mathbf{B}|/|\mathbf{A}|)\sin(angleAB) & 0 \end{bmatrix}$$

The cosine is the dot components of a vector divide

$$\cos(\mathbf{B},\mathbf{A}) = [\mathbf{B}/\bullet\mathbf{A}] = [\mathbf{B}/\mathbf{A}].dot() = \begin{bmatrix} (|\mathbf{B}|/|\mathbf{A}|)\cos(angleAB) & 0 \\ 0 & (|\mathbf{B}|/|\mathbf{A}|)\cos(angleAB) \end{bmatrix}$$

And finally, Tangent is sine divided by cosine.

$$\tan(\mathbf{B},\mathbf{A}) = \frac{[\mathbf{B}/\times\mathbf{A}]}{[\mathbf{B}/\bullet\mathbf{A}]} = \frac{[\mathbf{A}\times\mathbf{B}]}{[\mathbf{A}\bullet\mathbf{B}]} = \begin{bmatrix} 0 & -\tan(angleAB) \\ \tan(angleAB) & 0 \end{bmatrix}$$

These trigonometric functions are similar to legacy trigonometric function in the manner that the lengths of adjacent and opposite are a function of the length of the hypotenuse (B) (see Figure 3); however, these functions are superior to legacy trigonometric function because the result is an actual vector which contains the proper directions of the adjacent, opposite and tangent.

The strangeness is that the adjacent, opposite, hypotenuse (B), and tangents are found by right multiplying the trig matrix by the A vector. This is demonstrated in the following equations which are highlighted in Figure 3.

$$\text{opposite} = \sin(\mathbf{B},\mathbf{A})\mathbf{A}$$
$$\text{adjacent} = \cos(\mathbf{B},\mathbf{A})\mathbf{A}$$
$$\text{hypotenuse}=\mathbf{B} = [\sin(\mathbf{B},\mathbf{A}) + \cos(\mathbf{B},\mathbf{A})]\mathbf{A} = [\mathbf{B} / \mathbf{A}]\mathbf{A}$$
$$\text{tangent} = \tan(\mathbf{B},\mathbf{A})\mathbf{A}$$

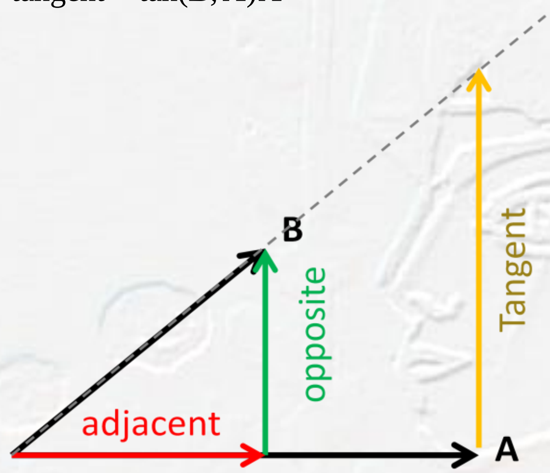

Figure 3: Sine, Cosine and tangent results from right multiply of A

By left multiplying the trig matrix by A results in reflected Sin, Hypotenuse and Tangent function as demonstrated by the following equations and Figure 4.

$$\text{opposite}' = \mathbf{A}\sin(\mathbf{B},\mathbf{A})$$
$$\text{adjacent}' = \mathbf{A}\cos(\mathbf{B},\mathbf{A})$$
$$\text{hypotenuse}'=|\mathbf{B}|\,\text{reflect}(\mathbf{B},\mathbf{A}) = \mathbf{A}[\sin(\mathbf{B},\mathbf{A}) + \cos(\mathbf{B},\mathbf{A})]\mathbf{A} = \mathbf{A}[\mathbf{B} / \mathbf{A}]$$
$$\text{tangent}' = \mathbf{A}\tan(\mathbf{B},\mathbf{A})$$

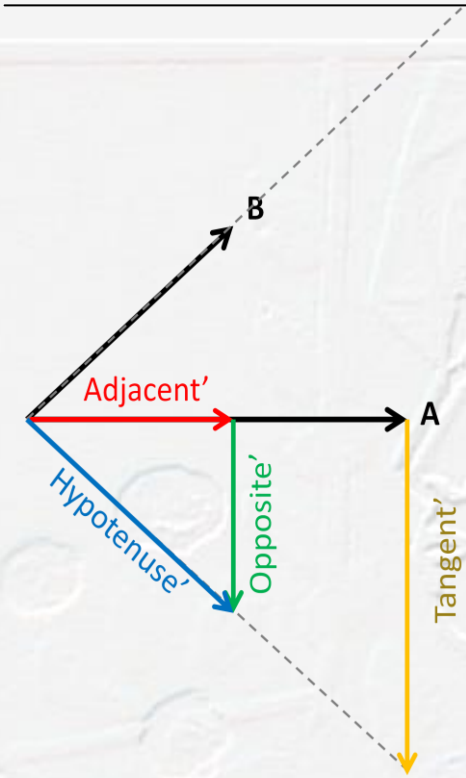

Figure 4: Reflected Trig results

Because left multiplication causes the Hypotenuse’ to appear as a reflection of B about A, these trigonometric results are called the reflected results. The reflected results are abbreviated using an apostrophe which results in Tangent’, Opposite’, Hypotenuse’, etc.

For completeness the results of multiplying the trig matrices by vector B are demonstrated next. The following equations and Figure 5 demonstrate the results of right multiplying the trig matrices by B. It should be noted that the magnitude of vector C is (B/A)B which means that if B is 50% of the size of A (close to what is shown in the diagrams) then C will be 50% the size of B, etc.

$$\text{opposite} = \sin(\mathbf{B},\mathbf{A})\mathbf{B}$$
$$\text{adjacent} = \cos(\mathbf{B},\mathbf{A})\mathbf{B}$$
$$\text{hypotenuse}=\mathbf{C} = [\sin(\mathbf{B},\mathbf{A}) + \cos(\mathbf{B},\mathbf{A})]\mathbf{B} = [\mathbf{B}/\mathbf{A}]\mathbf{B}$$
$$\text{tangent} = \tan(\mathbf{B},\mathbf{A})\mathbf{B}$$

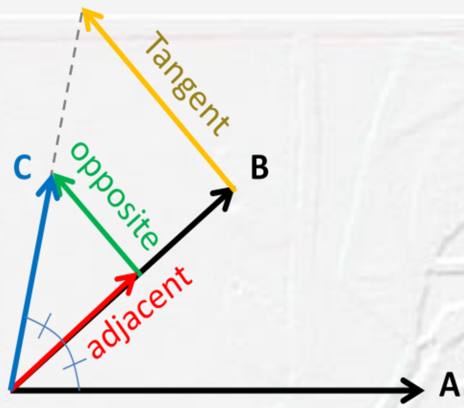

Figure 5: Trig Right Multiplied B

And finally, the last case is B left multiplied to the trig matrices as demonstrated by the following equations and Figure 6.

$$\text{opposite}' = \mathbf{B}\sin(\mathbf{B},\mathbf{A})$$
$$\text{adjacent}' = \mathbf{B}\cos(\mathbf{B},\mathbf{A})$$
$$\text{hypotenuse}' = \mathbf{C} = \mathbf{B}[\sin(\mathbf{B},\mathbf{A}) + \cos(\mathbf{B},\mathbf{A})] = \mathbf{B}[\mathbf{B}/\mathbf{A}]$$
$$\text{tangent}' = \mathbf{B}\tan(\mathbf{B},\mathbf{A})$$

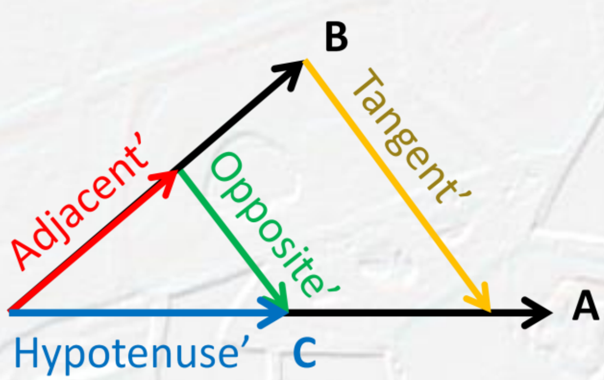

Figure 6: Trig Left Multiplied by B

This section demonstrates the various trigonometric results that are obtained from the various ways that multiplication can be performed. From these basic results the remaining trigonometric functions can be derived such as Cotangent , Secant, Cosecant, Exsecant, Excosecant, Versine, and Coversine.

This section further demonstrates the necessity of right and left operators. This gives validity to the notion that a proper vector product can neither be commutative or associative otherwise ambiguous directions would results.

4.17 Euler's Vortrix (2D)

Euler's Equation is

$$e^{\mathbf{i}\theta} = \cos(\theta) + \mathbf{i}\sin(\theta)$$

Substituting the Vortrix replacement for the complex operator yields

$$e^{\mathbf{i}\theta} = \cos(\theta) + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \sin(\theta) = \cos(\theta) + \begin{bmatrix} 0 & -\sin(\theta) \\ \sin(\theta) & 0 \end{bmatrix}$$

Since the cosine is a scalar and scalars can only exist as a scalar matrix then

$$e^{\mathbf{i}\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

And the conjugate is

$$e^{-\mathbf{i}\theta} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

The following derivation is applicable to either case above. Using the first case, multiplying by |A||B| and replacing θ with angleAB yields

$$|\mathbf{A}||\mathbf{B}|e^{\mathbf{i}(angleAB)} = \begin{bmatrix} |\mathbf{A}||\mathbf{B}|\cos(angleAB) & -|\mathbf{A}||\mathbf{B}|\sin(angleAB) \\ |\mathbf{A}||\mathbf{B}|\sin(angleAB) & |\mathbf{A}||\mathbf{B}|\cos(angleAB) \end{bmatrix}$$

$$[\mathbf{AB}] = |\mathbf{A}||\mathbf{B}|e^{\mathbf{i}(angleAB)}$$

Vortrix Algebra is superior to complex numbers because its matrix structure eliminates the need for imaginary constructs. All prior mathematical constructs that used imaginary constructs can now be replaced with real Vortrix constructs.

Continuing this train of thought, A and B are replaced with direction vectors

$$[\hat{\mathbf{A}}\hat{\mathbf{B}}] = e^{\mathbf{i}(angleAB)} = \begin{bmatrix} \cos(angleAB) & \sin(angleAB) \\ -\sin(angleAB) & \cos(angleAB) \end{bmatrix}$$

$$[\hat{\mathbf{A}}\times\hat{\mathbf{B}}]=[\sin(\hat{\mathbf{A}}\hat{\mathbf{B}})]=\mathbf{i}\sin(angleAB)=\begin{bmatrix}0 & -\sin(angleAB)\\ \sin(angleAB) & 0\end{bmatrix}$$

$$[\hat{\mathbf{A}}\bullet\hat{\mathbf{B}}]=[\cos(\hat{\mathbf{A}}\hat{\mathbf{B}})]=\cos(angleAB)=\begin{bmatrix}\cos(angleAB) & 0\\ 0 & \cos(angleAB)\end{bmatrix}$$

These results are identical to the forms derived in 4.16.4

4.18 Ln[AB]

<UNFINISHED>

Consider that

$$[\mathbf{AB}]=\begin{bmatrix}|\mathbf{A}||\mathbf{B}|\cos(angleAB) & -|\mathbf{A}||\mathbf{B}|\sin(angleAB)\\ |\mathbf{A}||\mathbf{B}|\sin(angleAB) & |\mathbf{A}||\mathbf{B}|\cos(angleAB)\end{bmatrix}$$

And

$$e^{\mathbf{i}(angleAB)}=\begin{bmatrix}\cos(angleAB) & -\sin(angleAB)\\ \sin(angleAB) & \cos(angleAB)\end{bmatrix}$$

Then

$$|\mathbf{A}||\mathbf{B}|e^{\mathbf{i}(angleAB)}=\begin{bmatrix}|\mathbf{A}||\mathbf{B}|\cos(angleAB) & -|\mathbf{A}||\mathbf{B}|\sin(angleAB)\\ |\mathbf{A}||\mathbf{B}|\sin(angleAB) & |\mathbf{A}||\mathbf{B}|\cos(angleAB)\end{bmatrix}$$

$$\ln\left(|\mathbf{A}||\mathbf{B}|e^{\mathbf{i}(angleAB)}\right)=\ln\left[\mathbf{AB}\right]$$

$$\ln(|\mathbf{A}|)+\ln(|\mathbf{B}|)+\mathbf{i}(angleAB)=\ln\left[\mathbf{AB}\right]$$

4.19 The Vortrix Natural Exponent $e^{[AB]}$

<UNFINISHED>

The infinite series expansion for the natural exponent is typically shown in its simplified form as follows.

$$e^x=1+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}\dots$$

The more complete representation is

$$e^{x}=\frac{x^{0}}{0!}+\frac{x^{1}}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}+\frac{x^{5}}{5!}\ldots$$

Replacing x with [AB]

$$e^{[\mathbf{AB}]}=\frac{[\mathbf{AB}]^{0}}{0!}+\frac{[\mathbf{AB}]^{1}}{1!}+\frac{[\mathbf{AB}]^{2}}{2!}+\frac{[\mathbf{AB}]^{3}}{3!}+\frac{[\mathbf{AB}]^{4}}{4!}+\frac{[\mathbf{AB}]^{5}}{5!}\ldots$$

For the sake of decomposing what this means, separate the magnitudes from the vectors

$$e^{(|\mathbf{A}||\mathbf{B}|)[\hat{\mathbf{A}}\hat{\mathbf{B}}]}=\frac{(|\mathbf{A}||\mathbf{B}|)[\hat{\mathbf{A}}\hat{\mathbf{B}}]^{0}}{0!}+\frac{(|\mathbf{A}||\mathbf{B}|)^{1}[\hat{\mathbf{A}}\hat{\mathbf{B}}]^{1}}{1!}+\frac{(|\mathbf{A}||\mathbf{B}|)^{2}[\hat{\mathbf{A}}\hat{\mathbf{B}}]^{2}}{2!}+\frac{(|\mathbf{A}||\mathbf{B}|)^{3}[\hat{\mathbf{A}}\hat{\mathbf{B}}]^{3}}{3!}+\frac{(|\mathbf{A}||\mathbf{B}|)^{4}[\hat{\mathbf{A}}\hat{\mathbf{B}}]^{4}}{4!}\ldots$$

Considering a simplified case where A and B are unit direction vectors

$$e^{[\hat{\mathbf{A}}\hat{\mathbf{B}}]}=\frac{[\hat{\mathbf{A}}\hat{\mathbf{B}}]^{0}}{0!}+\frac{[\hat{\mathbf{A}}\hat{\mathbf{B}}]^{1}}{1!}+\frac{[\hat{\mathbf{A}}\hat{\mathbf{B}}]^{2}}{2!}+\frac{[\hat{\mathbf{A}}\hat{\mathbf{B}}]^{3}}{3!}+\frac{[\hat{\mathbf{A}}\hat{\mathbf{B}}]^{4}}{4!}+\frac{[\hat{\mathbf{A}}\hat{\mathbf{B}}]^{5}}{5!}\ldots$$

4.20 Properties of the 3D Vortrix

<UNFINISHED>

The identities in the following section were tested exhaustively using a computer. Thus the identities are sound; however, there has been insufficient time to understand the full nature of the 3D system to understand exactly why it all works.

Some preliminary notions of the nature of the 3D system follow:

Since the matrix has the capability to rotate a vector multiplied against it, the characteristics of the components of rotation are as follows:

$\measuredangle XY=\operatorname{arctan2}(M1,M0)$ = Angle by which X components are rotated to Y

$\measuredangle XZ=\operatorname{arctan2}(M2,M0)$ = Angle by which X components are rotated to Z

$\measuredangle YZ=\operatorname{arctan2}(M3,M0)$ = Angle by which Y components are rotated to Z

The above are similar to Euler Angles, the exact relationship is ongoing work due to the fact that there are also rotations (or inflations) into volume space. The angles by which the dimensions are inflated into volume are as follows

$\measuredangle XV = \arctan 2(M3, M0)$ = Angle by which X is rotated to V

$\measuredangle YV = \arctan 2(-M2, M0)$ = Angle by which Y is rotated to V

$\measuredangle ZV = \arctan 2(M1, M0)$ = Angle by which Z is rotated to V

4.21 Rotations in 3D

<UNFINISHED>

5 Applications

5.1 Pythagorean Theory

Given three vectors A, B and C such that

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

Square both sides using Vortrix Algebra

$$\mathbf{C}^2 = (\mathbf{A} + \mathbf{B})^2$$

$$\mathbf{C}^2 = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B})$$

$$\mathbf{C}^2 = \mathbf{A}^2 + [\mathbf{AB}] + [\mathbf{BA}] + \mathbf{B}^2$$

$$\mathbf{C}^2 = \mathbf{A}^2 + [\mathbf{A} \bullet \mathbf{B}] + [\mathbf{A} \times \mathbf{B}] + [\mathbf{B} \bullet \mathbf{A}] + [\mathbf{B} \times \mathbf{A}] + \mathbf{B}^2$$

The dot products add, the cross products cancel (BxA = -AxB)

$$\mathbf{C}^2 = \mathbf{A}^2 + 2[\mathbf{A} \bullet \mathbf{B}] + \mathbf{B}^2$$

If A and B are perpendicular, then the dot product is zero and

$$\mathbf{C}^2 = \mathbf{A}^2 + \mathbf{B}^2$$

5.2 Vortrix Calculus Chain Rule

Given the product of two time varying vectors [**AB**] we should like to calculate the time derivative. Using the standard limit definition to determine the derivative, begin with

$$\frac{d}{dt}([\mathbf{AB}]) = \lim_{\Delta t \to 0}\left(\frac{(\mathbf{A}+\dot{\mathbf{A}}\Delta t)(\mathbf{B}+\dot{\mathbf{B}}\Delta t)-\mathbf{AB}}{\Delta t}\right)$$

Multiplying through

$$\frac{d}{dt}([\mathbf{AB}]) = \lim_{\Delta t \to 0}\left(\frac{[\mathbf{AB}]+[\mathbf{A}\dot{\mathbf{B}}]\Delta t+[\dot{\mathbf{A}}\mathbf{B}]\Delta t+[\dot{\mathbf{A}}\dot{\mathbf{B}}]\Delta t^2-[\mathbf{AB}]}{\Delta t}\right)$$

Reducing

$$\frac{d}{dt}([\mathbf{AB}]) = \lim_{\Delta t \to 0}\left([\mathbf{A}\dot{\mathbf{B}}]+[\dot{\mathbf{A}}\mathbf{B}]+[\dot{\mathbf{A}}\dot{\mathbf{B}}]\Delta t\right)$$

Taking the limit

$$\frac{d}{dt}([\mathbf{AB}]) = [\mathbf{A}\dot{\mathbf{B}}]+[\dot{\mathbf{A}}\mathbf{B}]$$

In this case, the Vortrix Calculus Chain rule follows the arithmetic calculus chain rule

5.3 The Pretonic Vector Ampere Field (revised 1.3)

Vortrix Algebra was developed to support research in electromagnetic physics. It has long been surmised by this author that electromagnetic fields at the electronic level (external to protons, electrons, and neutrons) seem to be the synthesis of a more fundamental field phenomenon called the Pretonic fields. This is similar to the notion that atoms are synthesized from more fundamental particles such as protons, electron, and neutrons; which are themselves synthesized by ever more fundamental particles. Therefore; it is logical that the field effects seen at the electronic level (outside electrons, protons and neutrons) are synthesized by more fundamental particles (called Pretons) whose Pretonic Field emissions, affected by Preton motion, synthesis the fields (including gravity) seen at the electronic level.

The purpose of this derivation is to show how a very simple field phenomenon can be "spun" into multiple effects that match experimentally with fields at the electronic level. Consider a simple particle called a Preton which emits a Pretonic Vector Ampere field **I**. The subscript S denotes that this is the source of the field that is being modeled. The preton has a charge of Qs coulomb charges and a vector velocity of **V**s.

$$\mathbf{I}_S = -Q_S\left([\mathbf{V}_S / \mathbf{r}]\right)\hat{\mathbf{r}}$$

The vector **r** is from the position of the preton to a position in space were the field is to be evaluated.

The vector ampere field does not directly couple to other pretons; it is the time derivative of the vector ampere field which couples to other pretons. This is shown in the next equation

$$\mathbf{F}_T = -K_M Q_S Q_T \frac{d}{dt}\left([\mathbf{V}_S / \mathbf{r}]\right)\hat{\mathbf{r}}$$

The subscript T refers to the "Target" preton which is the particle reacting to the field emitted by the source. Ft is the vector force acting on the target preton which has a charge of Qt. Km is the magnetic constant which 1e-7 and the vector **r** is the vector from the source to the target; found by subtracting the vector position of the source from the vector position of the target.

$$\mathbf{r} = \mathbf{P}_T - \mathbf{P}_S$$

The derivative of the expression within the parenthesis requires a chain rule. Plugging into the standard limit expression for derivatives

$$\frac{d}{dt}\left([\mathbf{V}_S / \mathbf{r}]\right) = \lim_{\Delta t \to 0}\left(\frac{\left(\mathbf{V}_S + \dot{\mathbf{V}}_S \Delta t\right) / \left(\mathbf{r} + \dot{\mathbf{r}}\Delta t\right) - \mathbf{V}_S / \mathbf{r}}{\Delta t}\right)$$

Right multiplying top and bottom by $\left(\mathbf{r} + \dot{\mathbf{r}}\Delta t\right)$ yields

$$\frac{d}{dt}\left([\mathbf{V}_S / \mathbf{r}]\right) = \lim_{\Delta t \to 0}\left(\frac{\left(\mathbf{V}_S + \dot{\mathbf{V}}_S \Delta t\right) - [\mathbf{V}_S / \mathbf{r}]\left(\mathbf{r} + \dot{\mathbf{r}}\Delta t\right)}{\Delta t\left(\mathbf{r} + \dot{\mathbf{r}}\Delta t\right)]}\right)$$

Reducing

$$\frac{d}{dt}\left([\mathbf{V}_S / \mathbf{r}]\right) = \lim_{\Delta t \to 0}\left(\frac{\mathbf{V}_S + \dot{\mathbf{V}}_S \Delta t - \mathbf{V}_S - [\mathbf{V}_S / \mathbf{r}]\left(\dot{\mathbf{r}}\Delta t\right)}{\mathbf{r}]\Delta t + \dot{\mathbf{r}}]\Delta t^2}\right)$$

Further Reducing

$$\frac{d}{dt}\left([\mathbf{V}_S / \mathbf{r}]\right) = \lim_{\Delta t \to 0}\left(\frac{\dot{\mathbf{V}}_S - [\mathbf{V}_S / \mathbf{r}]\dot{\mathbf{r}}}{\mathbf{r}] + \dot{\mathbf{r}}]\Delta t}\right)$$

Taking limit

$$\frac{d}{dt}([\mathbf{V}_S/\mathbf{r}]) = \frac{\dot{\mathbf{V}}_S - [\mathbf{V}_S/\mathbf{r}]\dot{\mathbf{r}}}{\mathbf{r}]}$$

$$\frac{d}{dt}([\mathbf{V}_S/\mathbf{r}]) = \left[\dot{\mathbf{V}}_S/\mathbf{r}\right] - ([\mathbf{V}_S/\mathbf{r}]\dot{\mathbf{r}})/\mathbf{r}$$

Substituting

$$\dot{\mathbf{r}} = \mathbf{V}_T - \mathbf{V}_S$$

And

$$\dot{\mathbf{V}}_S = \mathbf{a}_S$$

Yields

$$\frac{d}{dt}([\mathbf{V}_S/\mathbf{r}]) = [\mathbf{a}_S/\mathbf{r}] - ([\mathbf{V}_S/\mathbf{r}](\mathbf{V}_T - \mathbf{V}_S))/\mathbf{r}$$

Substitute back into main expression

$$\mathbf{F}_T = -K_M Q_S Q_T([\mathbf{a}_S/\mathbf{r}] - ([\mathbf{V}_S/\mathbf{r}](\mathbf{V}_T - \mathbf{V}_S))/\mathbf{r})\hat{\mathbf{r}}$$

Reducing

$$\mathbf{F}_T = -K_M Q_S Q_T(\frac{\mathbf{a}_S}{|\mathbf{r}|} - \frac{[\mathbf{V}_S/\mathbf{r}](\mathbf{V}_T - \mathbf{V}_S)}{|\mathbf{r}|})$$

$$\mathbf{F}_T = -K_M Q_S Q_T(\frac{\mathbf{a}_S}{|\mathbf{r}|} - \frac{[\hat{\mathbf{r}}\mathbf{V}_S](\mathbf{V}_T - \mathbf{V}_S)}{|\mathbf{r}|^2})$$

$$\mathbf{F}_T = K_M Q_S Q_T\left(-\frac{\mathbf{a}_S}{|\mathbf{r}|} + \frac{[\hat{\mathbf{r}}\mathbf{V}_S](\mathbf{V}_T - \mathbf{V}_S)}{|\mathbf{r}|^2}\right)$$

The first term in the parentheses is identical to New Induction which was discovered by this author over 20 years ago from experimental means. New Induction obtains identical answers to Faradays Law for mutual inductance experiments and is able to provide correct answers for self-inductance and open loop inductance (Dipole) experiments from which Faraday's Law provides no answer [Distinti 1]

The second term inside the parentheses is the candidate to supersede New Magnetism. New Magnetism is an amalgam derived from classical magnetic field models plus the addition of a term to account for a simple two wire experiment that legacy EM theory missed. As of this writing, limited experimental testing has produced no divergence; however, much more testing is needed.

The only legacy EM field effect not seen in the above is the Coulomb field. It is a simple matter to show the synthesis of the Coulomb field from the Preton Vector Ampere Field from a system of two or more Pretons. This derivation is not shown in this paper.

6 TBD

<UNFINSHED>

6.1 Sub heading

6.1.1 Sub sub heading

6.1.2 Second

6.1.3 Third

6.2 Sub two

6.3 Sub 3

6.3.1 Other

6.3.2

6.3.3

6.3.4

6.3.5

Appendix A. Consolidated List of Identities

<UNFINSHED>

Appendix B. Software Supplement

<UNFINSHED>

Appendix C. Identity Verification

Appendix D.

Appendix E.

Appendix F.

Appendix G.

